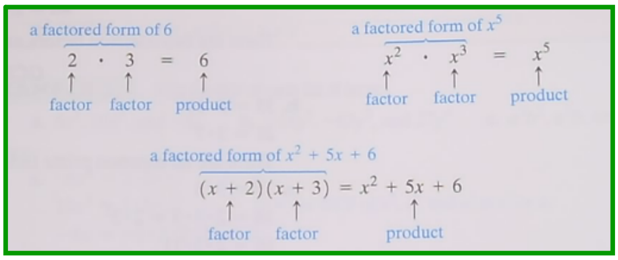
6.1 The Greatest Common Factor & Factoring by Grouping DAY ONE

REVIEW



Product (2)(3) = 6

2 & 3 are factors of 6

(2)(3) is in factored form of 6

The process of writing a polynomial as a product is called factoring the polynomial.

OBJECTIVE 1: Finding the Greatest Factor of a List of Integers

The first step in factoring a polynomial is to see whether the terms of the polynomial have a common factor (GCF).

If there is one then write the polynomial as a product by factoring out the common factor.

This term factored out is called the greatest common factor or GCF.

Finding the GCF of a List of Integers

Step 1. Write each number as a product of prime numbers.

Step 2. Identify the common prime factors.

Step 3. The product of all common prime factors found in Step 2 is the greatest common factor. If there are no common prime factors, the greatest common factor is 1.

Example 1: Find the GCF of each list of numbers.

a) 28 and 40 b) 55 and 21 c) 15, 18, and 66

Practice 1:

a) 36 and 42 b) 35 and 44 c) 12, 16, and 40

OBJECTIVE 2: Finding the Greatest Common Factor of a List of Terms

The greatest common factor of a list of variables raised to powers is found in a similar way.

$$x^2 = x \cdot x$$

$$x^3 = x \cdot x \cdot x$$

$$x^5 = x \cdot x \cdot x \cdot x \cdot x$$

Example 2: Find the GCF of each list of terms.

a) x^3, x^7, and x^5

b) y, y^4, and y^7

Practice 2:

a) y^6, y^4, and y^7

b) x, x^4, and x^2

Helpful Hint

Remember that the GCF of a list of terms contains the smallest exponent on each common variable.

The GCF of x^5y^6, x^2y^7, and x^3y^4 is x^2y^4. (Smallest exponent on x; Smallest exponent on y)

Example 3: Find the GCF of each list of terms.

a) $6x^2$, $10x^3$, & $-8x$

b) $18y^2$, $-63y^3$, & $27y^4$

c) a^3b^2, a^5b, & a^6b^2

Practice 3: Find the GCF of each list of terms.

a) $5y^4$, $15y^2$, & $-20y^3$

b) $4x^2$, x^3, & $3x^8$

c) a^4b^2, a^3b^5, & a^2b^3

OBJECTIVE 3: Factoring Out the Greatest Common Factor

First step is to factor out the GCF.

Helpful Hint

A factored form of $8x + 14$ is *not*

$$2 \cdot 4x + 2 \cdot 7$$

Although the *terms* have been factored (written as products), the *polynomial* $8x + 14$ has not been factored (written as a product).

A factored form of $8x + 14$ is the *product* $2(4x + 7)$.

Example 4: Factor each polynomial by factoring out the GCF.

a) $6t + 18$ b) $y^5 - y^7$

Practice 4:

a) $4t + 12$ b) $y^8 + y^4$

Example 5: Factor: $-9a^5 + 18a^2 - 3a$

Practice 5: $-8b^6 + 16b^4 - 3a$

Examples 6 - 8: Factor.

6) $6a^4 - 12a$

7) $\frac{3}{7}x^4 + \frac{1}{7}x^3 - \frac{5}{7}x^2$

8) $15p^2q^4 + 20p^3q^5 + 5p^3q^3$

Practices 6 - 8: Factor.

6) $5x^4 - 20x$

7) $\frac{3}{7}x^4 + \frac{1}{7}x^3 - \frac{5}{7}x^2$

8) $8a^2b^4 - 20a^3b^3 + 12ab^3$

6.1 Factoring by Grouping DAY TWO

OBJECTIVE 4: Factoring by Grouping

Once the GCF is factored out, we can often continue to factor the polynomial using a variety of techniques. We discuss here a technique for factoring polynomials called factoring by grouping.

Factoring by Grouping is most often used when there are four terms. The nice thing is it has a self built in checking system if done correctly.

Here are a few examples of the self built-in checking with factoring by grouping.

Example 1: Factor: $5(x + 3) + y(x + 3)$

Practice 1: $8(y - 2) + x(y - 2)$

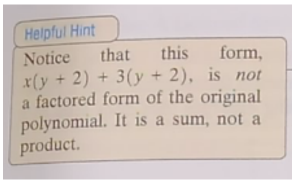
another example

Example 2: Factor: $3m^2n(a + b) - (a + b)$

Practice 2: $7xy^3(p + q) - (p + q)$

Whole process of factoring by grouping:

Example 3: Factor $xy + 2x + 3y + 6$


Helpful Hint

Notice that this form, $x(y + 2) + 3(y + 2)$, is *not* a factored form of the original polynomial. It is a sum, not a product.

Check by multiplying or FOILing!!

Practice 3: $xy + 3y + 4x + 12$

To Factor a Four-Term Polynomial by Grouping

Step 1. Group the terms in two groups of two terms so that each group has a common factor.

Step 2. Factor out the GCF from each group.

Step 3. If there is now a common binomial factor in the groups, factor it out.

Step 4. If not, rearrange the terms and try these steps again.

Examples 4 - 6: Factor by Grouping

4) $15x^3 - 10x^2 + 6x - 4$

5) $3x^2 + 4xy - 3x - 4y$

6) $2a^2 + 5ab + 2a + 5b$

Practices 4 - 6: Factor by Grouping

4) $40x^3 - 24x^2 + 15x - 9$

5) $3y^2 + 2xy - 2x - 3y$

6) $7a^3 + 5a^2 + 7a + 5$

Examples 7 & 8: Factor by grouping.
Sometimes we may need to change the order.

7) $3xy + 2 - 3x - 2y$ 8) $5x - 10 + x^3 - x^2$

Practice 7 & 8:

7) $4xy + 15 - 12x - 5y$ 8) $9y - 18 + y^3 - 4y^2$

Remember if you can first GCF, you should!

Example 9: Factor: $4ax - 4ab - 2bx + 2b^2$

Practice 9: $3xy - 3ay - 6ax + 6a^2$

> **Helpful Hint**
>
> Throughout this chapter, we will be factoring polynomials. Even when the instructions do not so state, it is always a good idea to check your answers by multiplying.

6.2 Factoring Trinomials of the Form $x^2 + bx + c$

OBJECTIVE 1: Factoring Trinomials of the Form $x^2 + bx + c$

These all have the coefficient of the squared variable is 1, ________. ______________ is the ________________ of ________. To check your factored answer, FOIL your answer.

Factoring a Trinomial of the Form $x^2 + bx + c$

The factored form of $x^2 + bx + c$ is

The product of these numbers is c.

$$x^2 + bx + c = (x + \square)(x + \square)$$

The sum of these numbers is b.

Example 1: Factor $x^2 + 7x + 12$

Practice 1: $x^2 + 5x + 6$

Example 2: Factor $x^2 - 17x + 70$

Practice 2: $x^2 - 17x + 70$

Example 3: Factor $x^2 + 4x - 12$

Practice: $x^2 + 5x - 14$

Example 4: Factor $r^2 - r - 42$

Practice 4: $p^2 - 2p - 63$

Example 5: Factor $a^2 + 2a + 10$

Practice 5: $b^2 + 5b + 1$

Example 6: Factor $x^2 + 7xy + 6y^2$

Practice 6: $x^2 + 7xy + 12y^2$

Example 7: Factor $x^4 + 5x^2 + 6$

Practice 7: $x^4 + 13x^2 + 12$

Example 8: Factor $40 - 13m + m^2$

Practice 8: $48 - 14x + x^2$

Helpful Hint

A positive constant in a trinomial tells us to look for two numbers with the same sign. The sign of the coefficient of the middle term tells us whether the signs are both positive or both negative.

both positive ↓ same sign ↙

$x^2 + 10x + 16 = (x + 2)(x + 8)$

both negative ↓ same sign ↙

$x^2 - 10x + 16 = (x - 2)(x - 8)$

A negative constant in a trinomial tells us to look for two numbers with opposite signs.

opposite signs ↓

$x^2 + 6x - 16 = (x + 8)(x - 2)$

opposite signs ↓

$x^2 - 6x - 16 = (x - 8)(x + 2)$

OBJECTIVE 2: Factoring Out the Greatest Common Factor (GCF)

Example 9: Factor $3m^2 - 24m - 60$

Practice 9: $4x^2 - 24x + 36$

Example 10: Factor $2x^4 - 26x^3 + 84x^2$

Practice 10: $3y^4 - 18y^3 - 21y^2$

6.3 Factoring Trinomials of the Form $ax^2 + bx + c$ & Perfect Square Trinomials DAY ONE

OBJECTIVE 1: Factoring Trinomials of the Form $ax^2 + bx + c$

Notice now the leading coefficient or your a something other than one. We will use an _____ method and factor by _________________ each time for consistency! It ALWAYS works!

Example 1: Factor $3x^2 + 11x + 6$

Practice 1: $2x^2 + 11x + 15$

Example 2: Factor $8x^2 - 22x + 5$

Practice 2: $15x^2 - 22x + 8$

Example 3: Factor $2x^2 + 13x - 7$

Practice 3: $4x^2 + 11x - 3$

Example 4: Factor $10x^2 - 13xy - 3y^2$

Practice 4: $21x^2 + 11xy - 2y^2$

Example 5: Factor $3x^4 - 5x^2 - 8$

Practice 5: $2x^4 - 5x^2 - 7$

Helpful Hint

Study the last two lines of Example 5. If a factoring attempt gives you a middle term whose numerical coefficient is the opposite of the desired numerical coefficient, try switching the signs of the last terms in the binomials.

Switched signs

$(3x^2 + 8)(x^2 - 1) = 3x^4 + 5x^2 - 8$ Middle term: $+5x^2$

$(3x^2 - 8)(x^2 + 1) = 3x^4 - 5x^2 - 8$ Middle term: $-5x^2$

OBJECTIVE 2: Factoring Out The Greatest Common Factor (GCF)

Don't forget that the ________________ in factoring ALWAYS is to look for a greatest common factor to factor out of ________________! Do not forget the ________ in the final factored form.

Example 6: Factor $24x^4 + 40x^3 + 6x^2$

Practice 6: $3x^3 + 17x^2 + 10x$

When "a" is negative, you may want to factor out a -1.

Example 7: Factor $-6x^2 - 13x + 5$

Practice 7: $-8x^2 + 2x + 3$

6.3 Perfect Square Trinomials DAY TWO

OBJECTIVE 3: Factoring Perfect Square Trinomials

A trinomial that is the square of a binomial is called a ______________________________.

$$(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9$$

6.3 DAY TWO

We were reminded of this in chapter 5 with special product formulas.

Factoring Perfect Square Trinomials

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

Helpful Hint

Notice that for both given forms of a perfect square trinomial, the last term is positive. This is because the last term is a square.

To use these we need to be able to recognize when we can. So, a trinomial is a perfect square when...

1) two terms, a^2 and b^2, are squared

2) the remaining term is (2)(a)(b) or (-2)(a)(b).

Example 8: Factor $x^2 + 12x + 36$

Practice 8: $x^2 + 14x + 49$

Example 9: Factor $25x^2 + 25xy + 4y^2$

Practice 9: $4x^2 + 20xy + 9y^2$

Example 10: Factor $4m^4 - 4m^2 + 1$

Practice 10: $36n^4 - 12n^2 + 1$

Example 11: Factor $162x^3 - 144x^2 + 32x$

Practice 11: $12x^3 - 84x^2 + 147x$

6.4 Factoring Trinomials of the Form $ax^2 + bx + c$ by Grouping

OBJECTIVE 1: Using the Grouping Method

This method will be used if you have four terms, or if you have a ≠ 1.

To Factor Trinomials by Grouping

Step 1. Factor out the greatest common factor if there is one other than 1.

Step 2. For the resulting trinomial $ax^2 + bx + c$, find two numbers whose product is $a \cdot c$ and whose sum is b.

Step 3. Write the middle term, bx, using the factors found in Step 2.

Step 4. Factor by grouping.

6.4

Example 1: Factor $3x^2 + 31x + 10$

Practice 1: $5x^2 + 61x + 12$

Example 2: Factor $8x^2 - 14x + 5$

Practice 2: $12x^2 - 19x + 5$

Example 3: Factor $6x^2 - 2x - 20$

Practice 3: $30x^2 - 14x - 4$

Example 4: Factor $18y^4 + 21y^3 - 60y^2$

Practice 4: $40m^4 + 5m^3 - 35m^2$

Example 5: Factor $4x^2 + 20x + 25$

Practice 5: $16x^2 + 24x + 9$

6.5 Factoring Binomials

OBJECTIVE 1: Factoring the Difference of Two Squares

Reminder of special products from Chapter 5.

$$(a + b)(a - b) = (a^2 - b^2)$$

This binomial is called a difference of squares.

Factoring the Difference of Two Squares

$$a^2 - b^2 = (a + b)(a - b)$$

6.5

Factoring the Difference of Two Squares

$$a^2 - b^2 = (a + b)(a - b)$$

Helpful Hint

Since multiplication is commutative, remember that the order of factors does not matter. In other words,

$$a^2 - b^2 = (a + b)(a - b) \text{ or } (a - b)(a + b)$$

Example 1: Factor $x^2 - 25$

Practice 1: $x^2 - 81$

Example 2: Factor each difference of squares.

a) $4x^2 - 1$ b) $25a^2 - 9b^2$ c) $y^2 - \frac{4}{9}$

Practice 2:

a) $9x^2 - 1$ b) $36a^2 - 49b^2$ c) $p^2 - \frac{25}{36}$

Example 3: Factor $x^4 - y^6$

Practice 3: $p^4 - q^{10}$

Example 4: Factor each binomial.

a) $y^4 - 16$ b) $x^2 + 4$

Practice 4:

a) $z^4 - 81$ b) $m^2 + 49$

Helpful Hint

When factoring, don't forget:

- See whether the terms have a greatest common factor (GCF) (other than 1) that can be factored out.
- Other than the GCF, the **sum** of two squares cannot be factored using real numbers.
- Factor completely. Always check to see whether any factors can be factored further.

Example 5 & 6: Factor each binomial.

5. $4x^3 - 49x$ 6. $162x^2 - 2$

Practice 5 & 6: Factor each binomial.

5. $36y^3 - 25y$

6. $80y^2 - 5$

Example 7: Factor $-49x^2 + 16$

Practice 7: $-9x^2 + 100$

OBJECTIVE 2: Factoring the Sum or Difference of Two Cubes

Sum of squares usually does not factor, but the sum and difference of cubes does work. The pattern leads to these two different formulas which I refer to as ______ to help remember the ______ of the formulas because the ____________________ ______________________________________!

Factoring the Sum or Difference of Two Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Remember "__________" means to "__________________ ________________"

Example 8: Factor $x^3 + 8$

Practice 8: $x^3 + 64$

SOAP

Helpful Hint

When factoring sums or differences of cubes, notice the sign patterns.

same sign

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

opposite signs always positive

same sign

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

opposite signs always positive

Example 9: Factor $y^3 - 27$

Practice 9: $x^3 - 125$

Example 10: Factor $64x^3 + 1$

Practice 10: $27y^3 + 1$

Example 11: Factor $54a^3 - 16b^3$

Practice 11: $32x^3 - 500y^3$

Graphing Calculator Explorations

TEACHING TIP

Some graphing calculators have a TABLE feature that allows the user to evaluate an expression for various values. Enter the expressions using the Y = key. Then use the TABLE feature.

Graphing

A graphing calculator is a convenient tool for evaluating an expression at a given replacement value. For example, let's evaluate $x^2 - 6x$ when $x = 2$. To do so, store the value 2 in the variable x and then enter and evaluate the algebraic expression.

```
2→X
                 2
X²-6X
                -8
```

The value of $x^2 - 6x$ when $x = 2$ is -8. You may want to use this method for evaluating expressions as you explore the following.

We can use a graphing calculator to explore factoring patterns numerically. Use your calculator to evaluate $x^2 - 2x + 1$, $x^2 - 2x - 1$, and $(x - 1)^2$ for each value of x given in the table. What do you observe?

	$x^2 - 2x + 1$	$x^2 - 2x - 1$	$(x - 1)^2$
$x = 5$			
$x = -3$			
$x = 2.7$			
$x = -12.1$			
$x = 0$			

Notice in each case that $x^2 - 2x - 1 \neq (x - 1)^2$. Because for each x in the table the value of $x^2 - 2x + 1$ and the value of $(x - 1)^2$ are the same, we might guess that $x^2 - 2x + 1 = (x - 1)^2$. We can verify our guess algebraically with multiplication:

$$(x - 1)(x - 1) = x^2 - x - x + 1 = x^2 - 2x + 1$$

6.6 Solving Quadratic Equations by Factoring

Quadratic Equation

A quadratic equation is one that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$.

The ________________ has a degree of two ALWAYS. In ____________ it is $ax^2 + bx + c = 0$. These equations can be very real world, like jumping off a cliff, punting a football, and many more.

OBJECTIVE 1: Solving Quadratic Equations by Factoring

Some quadratic equations can be solved by making use of factoring and the ________________________ (ZFP).

Zero Factor Property

If a and b are real numbers and if $ab = 0$, then $a = 0$ or $b = 0$.

This property states that if the product of two numbers is 0 then at least ______ of the numbers must be ____.

Example 1: Solve $(x - 3)(x + 1) = 0$.

Practice 1: $(x + 4)(x - 5) = 0$

Helpful Hint

The zero factor property says that *if a product is 0, then a factor is 0.*

If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

If $x(x + 5) = 0$, then $x = 0$ or $x + 5 = 0$.

If $(x + 7)(2x - 3) = 0$, then $x + 7 = 0$ or $2x - 3 = 0$.

Use this property only when the product is 0.

For example, if $a \cdot b = 8$, we do not know the value of a or b. The values may be $a = 2, b = 4$ or $a = 8, b = 1$, or any other two numbers whose product is 8.

Example 2: Solve $(x - 5)(2x + 7) = 0$

Practice 2: $(x - 12)(4x + 3) = 0$

Example 3: Solve $x(5x - 2) = 0$

Practice 3: $x(7x - 6) = 0$

Example 4: Solve $x^2 - 9x - 22 = 0$

Practice 4: $x^2 - 8x - 48 = 0$

Example 5: Solve $4x - 28x = -49$

Practice 5: $9x^2 - 24x = -16$

To Solve Quadratic Equations by Factoring

Step 1. Write the equation in standard form so that one side of the equation is 0.

Step 2. Factor the quadratic expression completely.

Step 3. Set each factor containing a variable equal to 0.

Step 4. Solve the resulting equations.

Step 5. Check each solution in the original equation.

Example 6: Solve $x(2x - 7) = 4$

Practice 6: $x(3x + 7) = 6$

Example 7: Solve $-2x^2 - 4x + 30 = 0$

Practice 7: $-3x^2 - 6x + 72 = 0$

OBJECTIVE 2: Solving Equations with Degree Greater Than Two by Factoring

Some equations involving polynomials of degree higher than 2 may also be solved by factoring and then applying the ZFP.

Example 8: Solve $3x^3 - 12x = 0$

Practice 8: $7x^3 - 63x = 0$

Example 9: Solve $(5x - 1)(2x^2 + 15x + 18) = 0$

Practice 9: $(3x - 2)(2x^2 - 13x + 15) = 0$

Example 10: Solve $2x^3 - 4x^2 - 30x = 0$

Practice 10: $5x^3 + 5x^2 - 30x = 0$

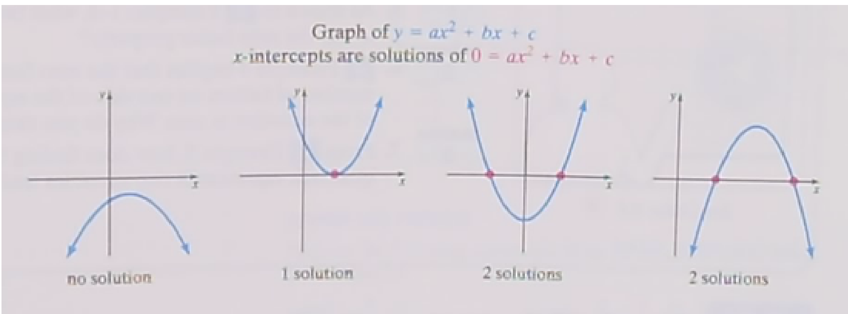
OBJECTIVE 3: Finding x-Intercepts of the Graph of a Quadratic Equation

Remember to find an x-intercept you need y to equal 0.

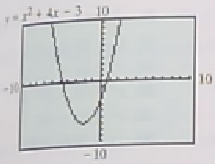
Example 11: Find the x-intercepts of the graph of $y = x^2 - 5x + 4$.

Practice 11: $y = x^2 - 6x + 8$

A quadratic equation in two variables is one that can be written in the form y = ax + bx +c where a 0. The graph is called a parabola and will open up (+ a) or down (- a) depending on the sign of a.

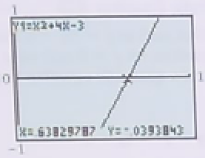


Graphing Calculator Explorations



A grapher may be used to find solutions of a quadratic equation whether the related quadratic polynomial is factorable or not. For example, let's use a grapher to approximate the solutions of $0 = x^2 + 4x - 3$. To do so, graph $y_1 = x^2 + 4x - 3$. Recall that the x-intercepts of this graph are the solutions of $0 = x^2 + 4x - 3$.

Notice that the graph appears to have an x-intercept between -5 and -4 and one between 0 and 1. Many graphers contain a TRACE feature. This feature activates a graph cursor that can be used to *trace* along a graph while the corresponding x- and y-coordinates are shown on the screen. Use the TRACE feature to confirm that x-intercepts lie between -5 and -4 and between 0 and 1. To approximate the x-intercepts to the nearest tenth, use a ROOT or a ZOOM feature on your grapher or redefine the viewing window. (A ROOT feature calculates the x-intercept. A ZOOM feature magnifies the viewing window around a specific location such as the graph cursor.) If we redefine the window to $[0, 1]$ on the x-axis and $[-1, 1]$ on the y-axis, the graph to the left is generated.



By using the TRACE feature, we can conclude that one x-intercept is approximately 0.6 to the nearest tenth. By repeating these steps for the other x-intercept, we find that it is approximately -4.6.

Use a grapher to approximate the real number solutions to the nearest tenth. If an equation has no real number solution, state so.

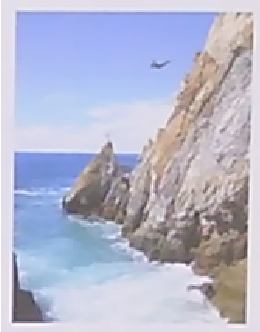
1. $3x^2 - 4x - 6 = 0$
2. $x^2 - x - 9 = 0$
3. $2x^2 + x + 2 = 0$
4. $-4x^2 - 5x - 4 = 0$
5. $-x^2 + x + 5 = 0$
6. $10x^2 + 6x - 3 = 0$

6.7 Quadratic Equations & Problem Solving

OBJECTIVE 1: Solving Problems Modeled by Quadratic Equations

Remember that these answers are real-world and therefore should ____________. For example, a person's age or the length of a rectangle is always a positive number. _______________________ that do not make sense as a solution to the problem.

Example 1: Finding Free-Fall Time



Since the 1940s, one of the top tourist attractions in Acapulco, Mexico, is watching the La Quebrada cliff divers. The divers' platform is about 144 feet above the sea. These divers must time their descent just right, since they land in the crashing Pacific Ocean in an inlet that is at most 9.5 feet deep. Neglecting air resistance, the height *h* in feet of a cliff diver above the ocean after *t* seconds is given by the quadratic equation $h = -16t^2 + 144$. Find how long it takes the diver to read the ocean.

Practice 1:

Cliff divers also frequent the falls at Waimea Falls Park in Oahu, Hawaii. One of the popular diving spots is 64 feet high. Neglecting air resistance, the height of a diver above the pool after *t* seconds is $h = -16t^2 + 64$. Find how long it takes a diver to reach the pool.

Example 2: Finding an Unknown Number

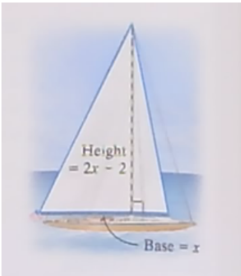
The square of a number plus three time the number is 70. Find the number.

Practice 2:

The square of a number minus eight times the number is equal to forty-eight. Find the number.

Example 3: Find the Dimensions of a Sail

The height of a triangular sail is 2 meters less than twice the length of the base. If the sail has an area of 30 square meters, find the length of its base and the height.



Practice 3:

An engineering team from Georgia Tech earned second place in a flight competition, with their triangular shaped paper hang glider. The base of their prize-winning entry was 1 foot less than three times the height. If the area of the triangular glider wing was 210 square feet, find the dimensions of the wing.

Example 4: Finding Consecutive Even Integers

Find two consecutive even integers whose product is 34 more than their sum.

Practice 4:

Find two consecutive integers whose product is 41 more than their sum.

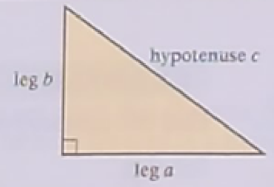
Helpful Hint

If you use this formula, don't forget that c represents the length of the hypotenuse.

Pythagorean Theorem

In a right triangle, the sum of the squares of the lengths of the two legs is equal to the square of the length of the hypotenuse.

$$(\text{leg})^2 + (\text{leg})^2 = (\text{hypotenuse})^2 \quad \text{or} \quad a^2 + b^2 = c^2$$



Example 5: Finding the Dimensions of a Triangle

Find the lengths of the sides of a right triangle if the lengths can be expressed as three consecutive even integers.

Practice 5:

Find the dimensions of a right triangle where the second leg is 1 unit less than double the first leg, and the hypotenuse is 1 unit more than double the length of the first leg.